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Global solvability, exponential decay and MFEM approximate solution of a nonlinear coupled system with moving boundary

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Abstract

In this work, we prove the existence and uniqueness of a strong regular solution for a certain class of a nonlinear coupled system of reaction-diffusion equations on a bounded domain with moving boundary. The exponential decay of the energy of the solutions, under the same assumptions, is also proved. In addition, we obtain approximate numerical solutions for systems of this type. In order to compare the theoretical and the numerical behaviour of the solutions, we

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develop a Matlab code based on the Moving Finite Element Method (MFEM) with high degree local approximations and we use an independent grid attached to each dependent variable. Finally, numerical tests which show the influence of the initial data on the exponential decay of the solution are performed.

Keywords: Nonlinear parabolic system, strong solution, moving boundary, nonlocal diffusion term, adaptive grids

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1. Introduction

In recent years, nonlinear parabolic partial differential equations (PDE) with nonlocal terms have been extensively studied by several authors such as Zheng [24], Chang [3], Corrêa [9] and Chipot [5, 6]. In particular, the existence, uniqueness and the exponential decay of strong global solutions for a class of nonlocal problems with moving boundaries is shown in [18]. The authors worked with a diffusion coefficient depending on the integral of the dependent variable over the time-dependent spatial domain. This type of diffusion coefficient, in a cylindrical domain, was initially proposed by Chipot and Lovat [4].

The most interesting real life problems involve more than one unknown function, so that to describe them a PDE system is necessary. References on systems of nonlinear equations with nonlocal terms are not so abundant in the literature. Raposo et al. [17], in 2008, studied the existence, uniqueness and exponential decay of solutions for reaction-diffusion coupled systems of the form

$$\begin{cases} u_t - a(l(u))\Delta u + f(u-v) = \alpha(u-v) & \text{in} \quad \Omega\times]0,T], \\ v_t - a(l(v))\Delta v - f(u-v) = \alpha(v-u) & \text{in} \quad \Omega\times]0,T], \end{cases}$$

with $a(\cdot) > 0$, l a continuous linear form, f a Lipschitz-continuous function and α a positive parameter. They also computed approximate solutions of these systems using implicit finite differences. Recently, Duque et al. [11] considered nonlinear systems of parabolic equations with a more general nonlocal diffusion term working on two linear forms l_1 and l_2:

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u = f_1(x,t) & \text{in} \quad \Omega\times]0,T], \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v = f_2(x,t) & \text{in} \quad \Omega\times]0,T]. \end{cases}$$

They improved the results obtained in [4, 9, 17] and also gave important results on polynomial and exponential decay, vanishing of the solutions in finite time and localization properties such as waiting time effect. The same authors proved, in [10], the convergence of a linearized Euler-Galerkin finite element method for the above problem and presented some numerical results. In [2], the authors investigated the propagation of an epidemic disease modeled by a system of three PDE, where the ith equation is of the type

$$(u_i)_t - a_i\left(\int_\Omega u_i dx\right)\Delta u_i = f_i\left(u_1, u_2, u_3\right),$$

in a physical domain $\Omega \subset \mathbb{R}^n$, $(n = 1, 2, 3)$. Santos et al. [20] established the exponential energy decay of the solutions for nonlinear coupled systems for beam equations with memory in noncylindrical domains. However, as in [11, 12, 13, 19], for example, they did not address the numerical analysis and simulation of the problem.

Our aim of this paper is to study the existence and uniqueness of a strong regular solutions for coupled systems of parabolic equations of the form:

$$\begin{cases} u_t - a_1 \left(\int_{\Omega_t} v(x,t)dx \right) u_{xx} = f_1(x,t) \ , & \text{in } \hat{Q}, \\ v_t - a_2 \left(\int_{\Omega_t} u(x,t)dx \right) v_{xx} = f_2(x,t) \ , & \text{in } \hat{Q}, \end{cases} \tag{1}$$

subject to the null Dirichlet boundary conditions, where $\hat{Q}$ is a connected bounded domain with lateral moving boundary. We generalize to systems of nonlinear equation the results obtained in [18] . To the best of our knowledge, none of the existing papers on solvability and stability for parabolic PDE systems deal with this type of diffusion in noncylindrical domains, so these results are the first in this direction. Moreover, we perform the numerical analysis directly in $\hat{Q}$.

In order to prove the existence of a solution to this problem, we consider a transformation τ, of class $\mathcal{C}^2$, from $\hat{Q}$ into a cylinder Q and establish the existence of a solution of the transformed problem in Q, by the Faedo-Galerkin method. Then, using the function τ^{-1}, we obtain the existence result for the primary problem. We also prove that, for this model, the energy is strong enough to produce exponential decay for the solution of the coupled system. In (1), the coupling occurs in the time-dependent diffusion coefficient a_i, which is determined by a global quantity. So, this system has nonlocal nonlinearity.

Problem (1) arises in a large class of real models, namely, in biology, where it could govern the spreading of two different species of populations that interact in a time-dependent medium (the spatial domain) through the functions a_i, $i = 1, 2$. The supply of being by external sources is denoted by f_1 and f_2. In this case the solution (u, v) describes the densities of the two populations. Since we consider an expanding spatial domain, as time increase, there is an enlarged area to compete and interact. So, it is reasonable to assume that the mobility inside the medium depends on how crowded it is. Hence, it makes sense that the diffusion rate of the population of species i depends on the entire population of species j throughout the spatial domain, with $j \neq i$, rather than on the local density. A related particular case of model (1) results from the assumption that the velocity of migration depends only on the global population in a subdomain of $\hat{Q}$.

A general approach for the numerical simulation of time dependent systems of PDEs is the use of finite elements. The MFEM, originally formulated by Miller [16], provides a way of solving parabolic moving problems by using finite elements on grids which are themselves time dependent. So, at each instant, the method determines both the numerical solution of the problem and the position

of the moving nodes. The use of local polynomial approximations of any degree was introduced by Sereno [21, 22] and developed by Coimbra [7, 8] to solve problems in spatial domains of $\mathbb{R}^2$ with fixed boundary. In this work, we apply a Matlab implementation which generalizes their numerical algorithms and is prepared to handle the moving boundaries.

This paper is organized as follows: in section 2, we present the formulation of the problem and the hypotheses on the data. In the two following sections, we use the Galerkin approximation, Aubin-Lions Theorem and the energy method introduced by Lions [15] to prove the global existence and uniqueness of strong solutions for coupled systems (1). The asymptotic behaviour of the global solutions for large t is investigated in Section 5. We perform a numerical study of problem (1) in Section 6: we introduce the relevant aspects of the MFEM and apply it to obtain an approximate numerical solution. To finalize this study, in Section 7, we draw some conclusions.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. In what follows, let $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$ be, respectively, the inner product and the norms in $L^2(\Omega)$ and $H_0^1(\Omega)$, given by

$$(u,v) = \int_\Omega u(x)v(x)dx \quad \text{and} \quad |u|^2 = \int_\Omega u^2 dx\,,$$
$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v dx \quad \text{and} \quad \|u\|^2 = \int_\Omega |\nabla u|^2 dx\,.$$

If X is a Banach space, we denote by $L^p(0,T;X)$, $1 \le p \le \infty$, the Banach space of vector valued functions $u :]0,T[\longrightarrow X$, which are measurable and $\|u(t)\|_X \in L^p(]0,T[)$, with the norms:

$$\|u(t)\|_{L^p(0,T;X)} = \left[\int_0^T \|u(t)\|_X^p dt\right]^{\frac{1}{p}}, \quad 1 \le p < \infty\,,$$
$$\|u(t)\|_{L^p(0,T;X)} = ess \sup_{0\le t<T} \|u(t)\|_X\,, \quad p = \infty\,.$$

In this work, we study the solutions of one-dimensional coupled systems with moving boundaries defined by

$$(\mathrm{P}_{u,v}) \begin{cases} u_t - a_1\left(\int_{\Omega_t} v(x,t)dx\right)u_{xx} = f_1(x,t)\,, & \text{for all } (x,t) \in \hat{Q}, \\ v_t - a_2\left(\int_{\Omega_t} u(x,t)dx\right)v_{xx} = f_2(x,t)\,, & \text{for all } (x,t) \in \hat{Q}, \\ u(\alpha(t),t) = u(\beta(t),t) = 0\,, & \text{for all } t \in]0,T[, \\ v(\alpha(t),t) = v(\beta(t),t) = 0\,, & \text{for all } t \in]0,T[, \\ u(x,0) = u_0(x)\,, \quad v(x,0) = v_0(x)\,, & x \in \Omega_0 =]\alpha(0),\beta(0)[, \end{cases} \tag{2}$$

where $\hat{Q}$ is a non-cylindrical domain of the plane $\mathbb{R}^2$, defined as follows:

$$\hat{Q} = \left\{(x,t) \in \mathbb{R}^2 : \ \alpha(t) < x < \beta(t), \ \text{ for all } 0 < t < T\right\} ,$$

where T is an arbitrary positive real number, $(\cdot)_t = \partial/\partial t$, $(\cdot)_{xx} = \partial^2/\partial x^2$ and a denotes a positive real continuous function. The lateral boundary of $\hat{Q}$ is given by $\hat{\Sigma} = \bigcup_{0<t<T} (\{\alpha(t), \beta(t)\} \times \{t\})$. Moreover, we assume that $\alpha'(t) < 0$ and $\beta'(t) > 0$, for all $t \in [0,T]$.

Note that the hypotheses $\alpha'(t) < 0$ and $\beta'(t) > 0$ imply that $\hat{Q}$ is increasing, in the sense that if $t_2 > t_1$, then the projection of $[\alpha(t_1), \beta(t_1)]$ onto the subspace $t = 0$ is contained in the projection of $[\alpha(t_2), \beta(t_2)]$ onto the same subspace. This also means that the real function $\gamma(t) = \beta(t) - \alpha(t)$ is increasing on $0 \le t < T$.

Observe that when (x,t) varies in $\hat{Q}$, the point (y,t) of $\mathbb{R}^2$, with $y = (x - \alpha(t))/\gamma(t)$, varies in the cylinder $Q =]0,1[\times]0,T[$. Thus, we have the function $\tau : \hat{Q} \longrightarrow Q$ given by $\tau(x,t) = (y,t)$, which is of class $\mathcal{C}^2$. The inverse τ^{-1} is also of class $\mathcal{C}^2$. The change of variable $\omega(y,t) = u(x,t)$, $\theta(y,t) = v(x,t)$, $g_1(y,t) = f_1(x,t)$ and $g_2(y,t) = f_2(x,t)$ with $x = \alpha(t) + \gamma(t)\, y$ transforms problem $(\mathrm{P}_{u,v})$ into problem $(\mathrm{P}_{\omega,\theta})$, given by

$$(\mathrm{P}_{\omega,\theta}) \begin{cases} \omega_t - b_1(y,t)\omega_y - a_1\left(\gamma(t)\displaystyle\int_0^1 \theta(y,t)dy\right) b_2(t)\omega_{yy} = g_1(y,t) \quad \text{in } Q, \\ \theta_t - b_1(y,t)\theta_y - a_2\left(\gamma(t)\displaystyle\int_0^1 \omega(y,t)dy\right) b_2(t)\theta_{yy} = g_2(y,t) \quad \text{in } Q, \\ \omega(0,t) = \omega(1,t) = 0 = \theta(0,t) = \theta(1,t)\,, \quad \text{for } 0 < t < T, \\ \omega(y,0) = \omega_0(y)\,, \quad \theta(y,0) = \theta_0(y)\,, \quad y \in]0,1[, \end{cases}$$

where $g_i(y,t) = f_i(\alpha + \gamma\, y, t)\,,\ i = 1,2$, $\omega_0(y) = u_0(\alpha(0) + \gamma(0)\, y)$ and $\theta_0(y) = v_0(\alpha(0) + \gamma(0)\, y)$. The coefficients $b_1(y,t)$ and $b_2(t)$ are defined by

$$b_1(y,t) = \frac{\alpha'(t) + \gamma'(t)y}{\gamma(t)} \quad \text{and} \quad b_2(t) = \frac{1}{(\gamma(t))^2}\,. \tag{3}$$

Since we are interested in proving the existence of a strong solution in $\hat{Q}$, let us consider the following hypotheses:

$(H1)$ $\alpha,\ \beta \in \mathcal{C}^2([0,T];\mathbb{R})$ and $0 < \gamma_0 < \gamma(t) < \gamma_1 < \infty$, for all $t \in [0,T]$,
$(H2)$ $\alpha',\ \beta' \in L^1(]0,T[) \cap L^2(]0,T[)$,
$(H3)$ $(u_0, v_0) \in H_0^1(\Omega_0) \times H_0^1(\Omega_0)\,, \quad \Omega_0 =]\alpha(0), \beta(0)[$,
$(H4)$ $(f_1, f_2) \in \left[L^2\left(0,T;L^2(\Omega_t)\right) \cap L^1\left(0,T;L^2(\Omega_t)\right)\right]^2\,,\ \Omega_t =]\alpha(t), \beta(t)[$,
$(H5)$ $a_i : \mathbb{R} \longrightarrow \mathbb{R}^+$ is lipschitz-continuous , $i = 1,2$
with $0 < m_{a_i} \le a_i(s) \le M_{a_i}$, for all $s \in \mathbb{R}$.

We may now state the main Theorem of this paper:

Theorem 1. *Under the assumptions $(H1) - (H5)$, there exists a unique strong solution $u, v : \hat{Q} \longrightarrow \mathbb{R}$ for problem $(P_{u,v})$, that is,*

$$\begin{aligned} u_t - a_1(l(v))\, u_{xx} &= f_1(x,t) \quad \text{in } L^2\left(0,T;L^2(\Omega_t)\right), \\ v_t - a_2(l(u))\, v_{xx} &= f_2(x,t) \quad \text{in } L^2\left(0,T;L^2(\Omega_t)\right), \end{aligned}$$

satisfying the regularity conditions:

$$\begin{aligned}&(u,v)\in\left[L^{\infty}\left(0,T;H_0^1\left(\Omega_t\right)\cap H^2\left(\Omega_t\right)\right)\right]^2,\\&(u_t,v_t)\in\left[L^2\left(0,T;L^2\left(\Omega_t\right)\right)\right]^2,\end{aligned}$$

where $l:L^2\left(\Omega_t\right)\longrightarrow\mathbb{R}$ *is a continuous linear form defined by* $l(\phi)=\int_{\alpha(t)}^{\beta(t)}\phi(x,t)dx$.

3. Existence of a solution for the transformed problem

In order to demonstrate the existence of a solution for the problem in Theorem 1, we first prove the existence of a solution for problem $(\mathrm{P}_{\omega,\theta})$ by applying the Faedo-Galerkin method, compactness argument and some technical ideas and then we use the diffeomorphism to establish the existence of a solution for the original problem. Consider the following hypotheses:

$$\begin{aligned}&(H3')\qquad(\omega_0,\theta_0)\in H_0^1\left(\Omega\right)\times H_0^1\left(\Omega\right),\\&(H4')\qquad(g_1,g_2)\in\left[L^2\left(0,T;L^2\left(\Omega\right)\right)\cap L^1\left(0,T;L^2\left(\Omega\right)\right)\right]^2,\end{aligned}$$

where $\Omega=]0,1[$.

Theorem 2. *Under the hypotheses* $(H1)-(H2)$, $(H3')-(H4')$ *and* $(H5)$, *there exists a solution* $\omega,\theta:Q\longrightarrow\mathbb{R}$ *of problem* $(P_{\omega,\theta})$, *that is,*

$$\begin{aligned}&\omega_t-b_1(y,t)\omega_y-a_1\left(l_1(\theta)\right)b_2(t)\omega_{yy}=g_1(y,t)\quad\text{in } L^2\left(0,T;L^2\left(\Omega\right)\right),\\&\theta_t-b_1(y,t)\theta_y-a_2\left(l_1(\omega)\right)b_2(t)\theta_{yy}=g_2(y,t)\quad\text{in } L^2\left(0,T;L^2\left(\Omega\right)\right),\end{aligned}$$

which satisfies the following conditions:

$$\begin{aligned}&(\omega,\theta)\in\left[L^{\infty}\left(0,T;H_0^1\left(\Omega\right)\cap H^2\left(\Omega\right)\right)\right]^2,\\&(\omega_t,\theta_t)\in\left[L^2\left(0,T;L^2\left(\Omega\right)\right)\right]^2,\end{aligned}$$

where the continuous linear form l_1 *is given by* $l_1(\chi)=\int_0^1\gamma(t)\,\chi(y,t)dy$.

Proof. We use the Faedo-Galerkin method to construct approximate solutions in a suitable finite dimensional space. Let $B=\{w_n(y)\}_{n\in\mathbb{N}}$ be a Hilbertian basis in $H_0^1(\Omega)$ and S_m be the subspace spanned by the first m vectors of B, that is $S_m=[w_1,w_2,...,w_m]$, $m=1,2,\dots$. Let us consider

$$\omega_m(t)=\sum_{i=1}^{m}c_{im}^{\omega}(t)w_i(y)\,;\;\;\theta_m(t)=\sum_{i=1}^{m}c_{im}^{\theta}(t)w_i(y)\,,\;\;0\le t<t_m\,,\;\;t_m<T\,.$$

We have that $(\omega_m(t),\theta_m(t))$ belongs to $S_m\times S_m$ and is the solution of the

system of ordinary differential equations

$$\begin{cases} \left(\dfrac{\partial \omega_m}{\partial t}, w\right) - \left(b_1 \dfrac{\partial \omega_m}{\partial y}, w\right) - a_1\left(l_1(\theta_m)\right) b_2 \left(\dfrac{\partial^2 \omega_m}{\partial y^2}, w\right) = (g_1, w)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } w \in S_m, \\ \left(\dfrac{\partial \theta_m}{\partial t}, w\right) - \left(b_1 \dfrac{\partial \theta_m}{\partial y}, w\right) - a_2\left(l_1(\omega_m)\right) b_2 \left(\dfrac{\partial^2 \theta_m}{\partial y^2}, w\right) = (g_2, w)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } w \in S_m, \\ \omega_m(0) = \omega_{0m} \to \omega_0 \quad \text{strongly in } H_0^1(\Omega), \\ \theta_m(0) = \theta_{0m} \to \theta_0 \quad \text{strongly in } H_0^1(\Omega), \end{cases}$$

which we will denote by $(\mathrm{P}^m_{\omega,\theta})$. As is well known, by Caratheodory's Theorem, problem $(\mathrm{P}^m_{\omega,\theta})$ has a local solution $(\omega_m(t), \theta_m(t))$ on some interval $[0, t_m[$, $0 < t_m < T$. The next a priori estimates permit to extension the solutions to the interval $[0, T]$ and to take limits in the approximate solutions of $(\mathrm{P}^m_{\omega,\theta})$.

We now deduce the a priori estimates which will be used in the proof of the Theorem.

Estimate I: Taking $w(t) = \omega_m(t)$ in the first equation of $(\mathrm{P}^m_{\omega,\theta})$ and $w(t) = \theta_m(t)$ in the second one, and taking into account that

$$\left(-\frac{\partial^2 \psi_m}{\partial y^2}, w\right) = \left(\frac{\partial \psi_m}{\partial y}, \frac{\partial w}{\partial y}\right), \qquad \text{for all } \psi_m, w \in S_m\,,$$

we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|\omega_m(t)\right|^2 - \left(b_1 \frac{\partial \omega_m}{\partial y}, \omega_m\right) + a_1\left(l_1(\theta_m)\right) b_2 \left|\frac{\partial \omega_m(t)}{\partial y}\right|^2 = (g_1, \omega_m)\,, \\ &\frac{1}{2}\frac{d}{dt}\left|\theta_m(t)\right|^2 - \left(b_1 \frac{\partial \theta_m}{\partial y}, \theta_m\right) + a_2\left(l_1(\omega_m)\right) b_2 \left|\frac{\partial \theta_m(t)}{\partial y}\right|^2 = (g_2, \theta_m)\,. \end{aligned} \tag{4}$$

Now, adding the two equations in (4) and using the equivalence of the norms in $H_0^1(\Omega)$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\left|\omega_m(t)\right|^2 + \left|\theta_m(t)\right|^2\right) - \left(b_1 \frac{\partial \omega_m}{\partial y}, \omega_m\right) - \left(b_1 \frac{\partial \theta_m}{\partial y}, \theta_m\right) \\ + a_1\left(l_1(\theta_m)\right) b_2 \left\|\omega_m(t)\right\|^2 + a_2\left(l_1(\omega_m)\right) b_2 \left\|\theta_m(t)\right\|^2 \\ = (g_1, \omega_m) + (g_2, \theta_m)\,. \end{aligned} \tag{5}$$

Integrating by parts the second and third terms in (5) and using the boundary conditions, we obtain

$$\begin{aligned} \int_0^1 b_1 \frac{\partial \psi_m}{\partial y} \psi_m dy = \int_0^1 \frac{b_1}{2}\frac{\partial}{\partial y}\left|\psi_m(y,t)\right|^2 dy = -\frac{1}{2}\int_0^1 \frac{\gamma'}{\gamma}\left|\psi_m(y,t)\right|^2 dy \\ = -\frac{1}{2}\frac{\gamma'}{\gamma}\left|\psi_m(t)\right|^2, \end{aligned} \tag{6}$$

for $\psi_m \in S_m$. Using the last identity and the Schwarz inequality in (5), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|\omega_m(t)|^2+|\theta_m(t)|^2\right)+\frac{1}{2}\frac{\gamma'}{\gamma}\left(|\omega_m(t)|^2+|\theta_m(t)|^2\right)\\ +m_a b_2\left(\|\omega_m(t)\|^2+\|\theta_m(t)\|^2\right)\leq\frac{1}{2}\left(|g_1(t)|^2+|\omega_m(t)|^2\right)\\ +\frac{1}{2}\left(|g_2(t)|^2+|\theta_m(t)|^2\right),\end{aligned} \tag{7}$$

where $m_a=\min\{m_{a_1},m_{a_2}\}$. Integrating from 0 to t, we get

$$\begin{aligned}&|\omega_m(t)|^2+|\theta_m(t)|^2+2m_a\int_0^t b_2\left(\|\omega_m(s)\|^2+\|\theta_m(s)\|^2\right)ds\\ &\leq\int_0^t\left(|g_1(s)|^2+|g_2(s)|^2\right)ds+\int_0^t\left(1+\frac{|\gamma'|}{|\gamma|}\right)\left(|\omega_m(s)|^2+|\theta_m(s)|^2\right)ds\\ &\qquad+|\omega_{0m}|^2+|\theta_{0m}|^2.\end{aligned}$$

From $(H1)$ and $(H2)$, it follows that $|\gamma'|/|\gamma|\leq c_0/\gamma_0$, where c_0 is independent of m. Let $c^*=\min\{1,\frac{2m_a}{\gamma_1^2}\}$. Then, using $(H4')$, $\omega_m(0)=\omega_{0m}\to\omega_0$ strongly in $H_0^1(\Omega)$, and $\theta_m(0)=\theta_{0m}\to\theta_0$ strongly in $H_0^1(\Omega)$. We then get

$$\begin{aligned}|\omega_m(t)|^2+|\theta_m(t)|^2+\int_0^t\left(\|\omega_m(s)\|^2+\|\theta_m(s)\|^2\right)ds\\ \leq C+C\int_0^t\left(|\omega_m(s)|^2+|\theta_m(s)|^2\right)ds,\end{aligned} \tag{8}$$

where $C=\max\{\frac{c_1}{c^*},\frac{1+\frac{c_0}{\gamma_0}}{c^*}\}$. Applying Gronwall's Lemma to the last inequality, we obtain

$$|\omega_m(t)|^2+|\theta_m(t)|^2\leq C_1 \tag{9}$$

and

$$\int_0^t\left(\|\omega_m(s)\|^2+\|\theta_m(s)\|^2\right)ds\leq C_2, \tag{10}$$

where C_1 and C_2 are positive constants, independent of m and t. From (9) and (10) it follows that

$$\begin{array}{lll}(\omega_m)\text{ and }(\theta_m) & \text{are bounded in} & L^\infty\left(0,T;L^2(\Omega)\right),\\ (\omega_m)\text{ and }(\theta_m) & \text{are bounded in} & L^2\left(0,T;H_0^1(\Omega)\right).\end{array} \tag{11}$$

Then we can extend the solution to the interval $[0,T]$. Two more estimates are required to pass to the limit when $m\to\infty$.

Estimate II: We want to estimate the derivatives $\partial\omega_m/\partial t$ and $\partial\theta_m/\partial t$. Multiplying by $w(t)=\partial\omega_m(t)/\partial t$ and $w(t)=\partial\theta_m(t)/\partial t$ in the first and second

equations of $(\mathrm{P}^m_{\omega,\theta})$, respectively, we obtain

$$\left(\frac{\partial\omega_m}{\partial t},\frac{\partial\omega_m}{\partial t}\right)-\left(b_1\frac{\partial\omega_m}{\partial y},\frac{\partial\omega_m}{\partial t}\right)+a_1\left(l_1(\theta_m)\right)b_2\left(\frac{\partial\omega_m}{\partial y},\frac{\partial^2\omega_m}{\partial y\partial t}\right)$$
$$=\left(g_1,\frac{\partial\omega_m}{\partial t}\right),$$
$$\left(\frac{\partial\theta_m}{\partial t},\frac{\partial\theta_m}{\partial t}\right)-\left(b_1\frac{\partial\theta_m}{\partial y},\frac{\partial\theta_m}{\partial t}\right)+a_2\left(l_1(\omega_m)\right)b_2\left(\frac{\partial\theta_m}{\partial y},\frac{\partial^2\theta_m}{\partial y\partial t}\right)$$
$$=\left(g_2,\frac{\partial\theta_m}{\partial t}\right).$$

Adding the two equations, we get

$$\begin{aligned}\left|\frac{\partial\omega_m(t)}{\partial t}\right|^2+\left|\frac{\partial\theta_m(t)}{\partial t}\right|^2-\left(b_1\frac{\partial\omega_m}{\partial y},\frac{\partial\omega_m}{\partial t}\right)-\left(b_1\frac{\partial\theta_m}{\partial y},\frac{\partial\theta_m}{\partial t}\right)\\ +a_1\left(l_1(\theta_m)\right)b_2\left(\frac{\partial\omega_m}{\partial y},\frac{\partial^2\omega_m}{\partial y\partial t}\right)+a_2\left(l_1(\omega_m)\right)b_2\left(\frac{\partial\theta_m}{\partial y},\frac{\partial^2\theta_m}{\partial y\partial t}\right)\\ =\left(g_1,\frac{\partial\omega_m}{\partial t}\right)+\left(g_2,\frac{\partial\theta_m}{\partial t}\right).\end{aligned}\tag{12}$$

The third and fourth terms in (12) can be estimated as follows:

$$\begin{aligned}\left|\left(b_1\frac{\partial\psi_m}{\partial y},\frac{\partial\psi_m}{\partial t}\right)\right|&\le|b_1|\left|\frac{\partial\psi_m(t)}{\partial y}\right|\left|\frac{\partial\psi_m(t)}{\partial t}\right|\\ &\le\frac{|\alpha'|+|\gamma'|}{\gamma_0}\left\|\psi_m(t)\right\|\left|\frac{\partial\psi_m(t)}{\partial t}\right|,\end{aligned}\tag{13}$$

using the Schwarz inequality and hypothesis $(H1)$. Each one of the last two terms on the left hand side of equation (12) yields the inequality

$$\begin{aligned}a_i\left(l_1(\psi_m)\right)b_2\left(\frac{\partial\psi_m}{\partial y},\frac{\partial^2\psi_m}{\partial y\partial t}\right)&=a_i\left(l_1(v_m)\right)b_2\frac{1}{2}\frac{d}{dt}\left|\frac{\partial\psi_m(t)}{\partial y}\right|^2\\ &\ge\frac{m_{a_i}}{\gamma_1^2}\frac{1}{2}\frac{d}{dt}\left|\frac{\partial\psi_m(t)}{\partial y}\right|^2,\end{aligned}\tag{14}$$

for $\psi_m\in S_m$. Substituting (13) and (14), with $\psi_m=\omega_m$ and $\psi_m=\theta_m$, in (12),

we obtain

$$\left|\frac{\partial\omega_m(t)}{\partial t}\right|^2+\left|\frac{\partial\theta_m(t)}{\partial t}\right|^2+\frac{m_a}{\gamma_1^2}\frac{1}{2}\frac{d}{dt}\left(\left|\frac{\partial\omega_m(t)}{\partial y}\right|^2+\left|\frac{\partial\theta_m(t)}{\partial y}\right|^2\right)$$
$$\leq\frac{|\alpha'|+|\gamma'|}{\gamma_0}\left\|\omega_m(t)\right\|\left|\frac{\partial\omega_m(t)}{\partial t}\right|+\frac{|\alpha'|+|\gamma'|}{\gamma_0}\left\|\theta_m(t)\right\|\left|\frac{\partial\theta_m(t)}{\partial t}\right|$$
$$+\frac{1}{2}\left(|g_1(t)|^2+\left|\frac{\partial\omega_m(t)}{\partial t}\right|^2\right)+\frac{1}{2}\left(|g_2(t)|^2+\left|\frac{\partial\theta_m(t)}{\partial t}\right|^2\right). \tag{15}$$

We now apply the Young inequality to the first two terms of the right hand side of (15) and we obtain

$$\frac{1}{4}\left(\left|\frac{\partial\omega_m(t)}{\partial t}\right|^2+\left|\frac{\partial\theta_m(t)}{\partial t}\right|^2\right)+\frac{m_a}{\gamma_1^2}\frac{1}{2}\frac{d}{dt}\left(\left\|\omega_m(t)\right\|^2+\left\|\theta_m(t)\right\|^2\right)$$
$$\leq\left(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\right)^2\left(\left\|\omega_m(t)\right\|^2+\left\|\theta_m(t)\right\|^2\right)+\frac{1}{2}\left(|g_1(t)|^2+|g_2(t)|^2\right).$$

Integrating from 0 to t, we have

$$\int_0^t\left(\left|\frac{\partial\omega_m(s)}{\partial s}\right|^2+\left|\frac{\partial\theta_m(s)}{\partial s}\right|^2\right)ds+\frac{2m_a}{\gamma_1^2}\left(\left\|\omega_m(t)\right\|^2+\left\|\theta_m(t)\right\|^2\right)$$
$$\leq\frac{4}{\gamma_0^2}\int_0^t\left(|\alpha'|+|\gamma'|\right)^2\left(\left\|\omega_m(s)\right\|^2+\left\|\theta_m(s)\right\|^2\right)ds$$
$$+2\int_0^t\left(|g_1(s)|^2+|g_2(s)|^2\right)ds+\frac{2m_a}{\gamma_1^2}\left(\left\|\omega_{0m}\right\|^2+\left\|\theta_{0m}\right\|^2\right).$$

From $(H2)$, we can ensure that

$$\int_0^t\left(|\alpha'|+|\gamma'|\right)^2\left\|\psi_m(s)\right\|^2ds\leq c_1\int_0^t\left\|\psi_m(s)\right\|^2ds\,,\quad\text{for }\psi_m\in S_m\,.$$

Therefore, in analogy to what we did for the first estimate, there is a positive constant C which does not depend on t and m, such that

$$\int_0^t\left(\left|\frac{\partial\omega_m(s)}{\partial s}\right|^2+\left|\frac{\partial\theta_m(s)}{\partial s}\right|^2\right)ds+\left\|\omega_m(t)\right\|^2+\left\|\theta_m(t)\right\|^2$$
$$\leq C+C\int_0^t\left(\left\|\omega_m(s)\right\|^2+\left\|\theta_m(s)\right\|^2\right)ds\,.$$

The Gronwall inequality yields

$$\left\|\omega_m(t)\right\|^2+\left\|\theta_m(t)\right\|^2\leq C_1, \tag{16}$$

and it follows that

$$\int_0^t \left(\left| \frac{\partial \omega_m(s)}{\partial s} \right|^2 + \left| \frac{\partial \theta_m(s)}{\partial s} \right|^2 \right) ds \leq C_2. \tag{17}$$

Finally,

$$\begin{array}{lll} (\omega_m) \text{ and } (\theta_m) & \text{are bounded in} & L^\infty\left(0,T;H_0^1(\Omega)\right), \\ (\frac{\partial \omega_m}{\partial t}) \text{ and } (\frac{\partial \theta_m}{\partial t}) & \text{are bounded in} & L^2\left(0,T;L^2(\Omega)\right). \end{array} \tag{18}$$

Estimate III: Setting $w(t) = -\partial^2\omega_m(t)/\partial y^2$ in the first equation of $(\mathrm{P}^m_{\omega,\theta})$ and $w(t) = -\partial^2\theta_m(t)/\partial y^2$ in the second one, and integrating the first term by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\omega_m(t)\|^2 - \left(b_1\frac{\partial \omega_m}{\partial y}, -\frac{\partial^2\omega_m}{\partial y^2}\right) - a_1\left(l_1(\theta_m)\right) b_2 \left(\frac{\partial^2\omega_m}{\partial y^2}, -\frac{\partial^2\omega_m}{\partial y^2}\right) \\ &\qquad\qquad = \left(g_1, -\frac{\partial^2\omega_m}{\partial y^2}\right), \\ &\frac{1}{2}\frac{d}{dt}\|\theta_m(t)\|^2 - \left(b_1\frac{\partial \theta_m}{\partial y}, -\frac{\partial^2\theta_m}{\partial y^2}\right) - a_2\left(l_1(\omega_m)\right) b_2 \left(\frac{\partial^2\theta_m}{\partial y^2}, -\frac{\partial^2\theta_m}{\partial y^2}\right) \\ &\qquad\qquad = \left(g_2, -\frac{\partial^2\theta_m}{\partial y^2}\right). \end{aligned}$$

Adding these two equations, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\omega_m(t)\|^2 + \|\theta_m(t)\|^2\right) + \left(b_1\frac{\partial \omega_m}{\partial y}, \frac{\partial^2\omega_m}{\partial y^2}\right) + \left(b_1\frac{\partial \theta_m}{\partial y}, \frac{\partial^2\theta_m}{\partial y^2}\right) \\ &\qquad + a_1\left(l_1(\theta_m)\right) b_2 \left|\frac{\partial^2\omega_m(t)}{\partial y^2}\right|^2 + a_2\left(l_1(\omega_m)\right) b_2 \left|\frac{\partial^2\theta_m(t)}{\partial y^2}\right|^2 \\ &\qquad = \left(g_1, -\frac{\partial^2\omega_m}{\partial y^2}\right) + \left(g_2, -\frac{\partial^2\theta_m}{\partial y^2}\right). \end{aligned} \tag{19}$$

Applying the arguments used in *Estimate II*, we obtain

$$\begin{aligned} \left(b_1\frac{\partial \psi_m}{\partial y}, \frac{\partial^2\psi_m}{\partial y^2}\right) &\leq \frac{|\alpha'| + |\gamma'|}{\gamma_0}\|\psi_m(t)\| \left|\frac{\partial^2\psi_m(t)}{\partial y^2}\right| \\ &\leq \frac{\varepsilon}{2}\left(\frac{|\alpha'| + |\gamma'|}{\gamma_0}\right)^2 \|\psi_m(t)\|^2 + \frac{1}{2\varepsilon}\left|\frac{\partial^2\psi_m(t)}{\partial y^2}\right|^2, \end{aligned} \tag{20}$$

for all $\varepsilon > 0$ and $\psi_m \in S_m$. The fourth term of the equation in (19) implies that

$$a_1\left(l_1(\theta_m)\right) b_2 \left|\frac{\partial^2\omega_m(t)}{\partial y^2}\right|^2 \geq \frac{m_{a_1}}{\gamma_1^2}\left|\frac{\partial^2\omega_m(t)}{\partial y^2}\right|^2, \tag{21}$$

and from the first term on the right hand side of (19), it follows that

$$\left| -\left(g_2, \frac{\partial^2\omega_m}{\partial y^2}\right)\right| \leq |g_2(t)| \left|\frac{\partial^2\omega_m(t)}{\partial y^2}\right| \leq \frac{\varepsilon}{2}|g_2(t)|^2 + \frac{1}{2\varepsilon}\left|\frac{\partial^2\omega_m(t)}{\partial y^2}\right|^2. \tag{22}$$

Note that we have inequalities similar to those in (21) and (22) for the function θ_m. Using these inequalities for θ_m, (21), (22) and substituting (20) with $\psi_m = \omega_m$ and $\psi_m = \theta_m$ in equation (19), we obtain

$$\begin{aligned}\frac{d}{dt}\left(\|\omega_m(t)\|^2 + \|\theta_m(t)\|^2\right) + 2\left(\frac{m_a}{\gamma_1^2} - \frac{1}{\varepsilon}\right)\left(\left|\frac{\partial^2\omega_m(t)}{\partial y^2}\right|^2 + \left|\frac{\partial^2\theta_m(t)}{\partial y^2}\right|^2\right)\\ \leq \varepsilon\left(\frac{|\alpha'| + |\gamma'|}{\gamma_0}\right)^2\left(\|\omega_m(t)\|^2 + \|\theta_m(t)\|^2\right) + \varepsilon\left(|g_1(t)|^2 + |g_2(t)|^2\right),\end{aligned} \tag{23}$$

where $m_a = \min\{m_{a_1}, m_{a_2}\}$. Observe that, for $\varepsilon > \gamma_1^2/m_a$, one has

$$\frac{m_a}{\gamma_1^2} - \frac{1}{\varepsilon} > 0\,.$$

So, set $\varepsilon = 2\gamma_1^2/m_a$, for example. Then, integrating from 0 to t, we obtain

$$\begin{aligned}\|\omega_m(t)\|^2 + \|\theta_m(t)\|^2 + \frac{m_a}{\gamma_1^2}\int_0^t\left(\left|\frac{\partial^2\omega_m(s)}{\partial y^2}\right|^2 + \left|\frac{\partial^2\theta_m(s)}{\partial y^2}\right|^2\right)ds\\ \leq \frac{2\gamma_1^2}{m_a}\left[\int_0^t\left(\frac{|\alpha'| + |\gamma'|}{\gamma_0}\right)^2\left(\|\omega_m(s)\|^2 + \|\theta_m(s\|^2\right)ds + \right.\\ \left. + \int_0^t |g_1(s)|^2 + |g_2(s)|^2\,ds\right] + \|\omega_{0m}\|^2 + \|\theta_{0m}\|^2\,.\end{aligned}$$

Using $(H4')$ and the strong convergencies in $H_0^1(\Omega)$, $\omega_m(0) = \omega_{0m} \to \omega_0$ and $\theta_m(0) = \theta_{0m} \to \theta_0$, we have that

$$\begin{aligned}\|\omega_m(t)\|^2 + \|\theta_m(t)\|^2 + \int_0^t\left(\left|\frac{\partial^2\omega_m(s)}{\partial y^2}\right|^2 + \left|\frac{\partial^2\theta_m(s)}{\partial y^2}\right|^2\right)ds\\ \leq C + C\int_0^t \|\omega_m(s)\|^2 + \|\theta_m(s)\|^2\,ds,\end{aligned} \tag{24}$$

where C is a positive constant that does not depend on t and m. By Gronwall's inequality and in analogy to what we did for the first two estimates, we can conclude that

$$\left(\tfrac{\partial^2\omega_m}{\partial y^2}\right) \text{ and } \left(\tfrac{\partial^2\theta_m}{\partial y^2}\right) \quad \text{are bounded in} \quad L^2\left(0,T;L^2(\Omega)\right)\ . \tag{25}$$

From the estimates obtained in (11), (18) and (25), we can extract subsequences of (ω_m) and (θ_m), which we still denote by (ω_m) and (θ_m), respectively, such that

$$\begin{aligned}&\omega_m \rightharpoonup \omega \quad \text{and} \quad \theta_m \rightharpoonup \theta \quad \text{in } L^2\left(0,T;H_0^1(\Omega)\right),\\ &\omega_m \overset{\star}{\rightharpoonup} \omega \quad \text{and} \quad \theta_m \overset{\star}{\rightharpoonup} \theta \quad \text{in } L^\infty\left(0,T;H_0^1(\Omega)\right),\\ &\frac{\partial\omega_m}{\partial t} \rightharpoonup \frac{\partial\omega}{\partial t} \quad \text{and} \quad \frac{\partial\theta_m}{\partial t} \rightharpoonup \frac{\partial\theta}{\partial t} \quad \text{in } L^2\left(0,T;L^2(\Omega)\right),\\ &\frac{\partial^2\omega_m}{\partial y^2} \rightharpoonup \frac{\partial^2\omega}{\partial y^2} \quad \text{and} \quad \frac{\partial^2\theta_m}{\partial y^2} \rightharpoonup \frac{\partial^2\theta}{\partial y^2} \quad \text{in } L^2\left(0,T;L^2(\Omega)\right).\end{aligned} \tag{26}$$

From the Aubin-Lions Compactness Lemma (see [15]), as $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) = \left(L^2(\Omega)\right)' \hookrightarrow L^2(\Omega)$, we have

$$\omega_m \to \omega \quad \text{and} \quad \theta_m \to \theta \quad \text{in} \quad L^2\left(0,T;L^2(\Omega)\right) . \tag{27}$$

Hence, passing if necessary to a subsequence (still denoted by the same symbol), one has

$$\omega_m \to \omega \quad \text{and} \quad \theta_m \to \theta \quad \text{a.e. in} \quad \Omega\times]0,T[. \tag{28}$$

Now we pass to the limit as $m \to \infty$. To pass to the limit in the nonlinear part, it is required to prove that

$$a_1\left(l_1\left(\theta_m\right)\right) \to a_1\left(l_1(\theta)\right) \quad \text{in } L^2(]0,T[) . \tag{29}$$

Since function a_1 is continuous by $(H5)$, it is sufficient to check that $l_1\left(\theta_m\right) - l_1(\theta) \to 0$ in $L^2(]0,T[)$, for each fixed t. In fact, we have that

$$\int_0^T \left|l_1\left(\theta_m\right) - l_1(\theta)\right|^2 dt = \int_0^T \left|l_1\left(\theta_m - \theta\right)\right|^2 dt \leq C \int_0^T \left|\theta_m - \theta\right|^2 dt < \epsilon .$$

The last inequality follows from the second convergency in (27). Applying the same arguments, we obtain

$$a_2\left(l_1\left(\omega_m\right)\right) \to a_2\left(l_1(\omega)\right) \text{ in } L^2(]0,T[) . \tag{30}$$

Using the convergencies in (26), (29) and (30), we can pass to the limit as $m \to \infty$ in the approximate problem $(\mathrm{P}^m_{\omega,\theta})$ and obtain the following :

$$\frac{\partial \omega}{\partial t} - b_1(y,t)\frac{\partial \omega}{\partial y} - a_1\left(l_1(\theta)\right) b_2(t)\frac{\partial^2 \omega}{\partial y^2} = g_1(y,t) \quad \text{em } L^2\left(0,T;L^2\left(\Omega\right)\right) ,$$
$$\frac{\partial \theta}{\partial t} - b_1(y,t)\frac{\partial \theta}{\partial y} - a_2\left(l_1(\omega)\right) b_2(t)\frac{\partial^2 \theta}{\partial y^2} = g_2(y,t) \quad \text{em } L^2\left(0,T;L^2\left(\Omega\right)\right) .$$

Now, we will verify the initial conditions. In fact, using the regularity result we have

$$\omega, \theta \in \mathcal{C}^0\left(0,T;L^2(\Omega)\right) .$$

In this manner, it makes sense to calculate $\omega(0)$ and $\theta(0)$. Let us consider $z \in \mathcal{C}^1\left(0,T;\mathbb{R}\right)$, with $z(0) = 1$ and $z(T) = 0$. Since the third convergency in (26) implies that

$$\int_0^T \left(\frac{\partial \omega_m}{\partial t}, \vartheta\right) z dt \to \int_0^T \left(\frac{\partial \omega}{\partial t}, \vartheta\right) z dt , \quad \vartheta \in L^2(\Omega) ,$$

performing integration by parts, we obtain

$$-\left(\omega_m(0), \vartheta\right) - \int_0^T \left(\omega_m, \vartheta\right) \frac{\partial z}{\partial t} dt \to -\left(\omega(0), \vartheta\right) - \int_0^T \left(\omega, \vartheta\right) \frac{\partial z}{\partial t} dt . \tag{31}$$

From (31) and using the second convergency in (26), we get $(\omega_m(0), \vartheta) \to (\omega(0), \vartheta)$, for all $\vartheta \in H_0^1(\Omega)$. But $\omega_m(0)$ converges strongly to ω_0 in $L^2(\Omega)$, and consequently weakly in $L^2(\Omega)$. Therefore, $(\omega_m(0), \vartheta) \to (\omega_0, \vartheta)$, for all $\vartheta \in H_0^1(\Omega)$. From the uniqueness of the limit, $(\omega(0), \vartheta) \to (\omega_0, \vartheta)$, for all $\vartheta \in H_0^1(\Omega)$. Thus $\omega(0) = \omega_0$. Similarly, we conclude that $\theta(0) = \theta_0$. Hence problem $(\mathrm{P}_{\omega,\theta})$ has a solution. □

In the next section, we address the uniqueness of the solution of $(\mathrm{P}_{\omega,\theta})$.

4. Uniqueness of the solution

The uniqueness of the global strong solution for the transformed problem with fixed boundaries is guaranteed by the following theorem.

Theorem 3. *Let $\omega, \theta : Q \longrightarrow \mathbb{R}$ be a global strong solution of $(P_{\omega,\theta})$ given by Theorem (2), $(\omega_0, \theta_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $0 < T < \infty$. Suppose that function a_i is Lipschitzian with constant $A_i > 0$, for $i = 1, 2$, that is,*

$$|a_i(s_1) - a_i(s_2)| \leq A_i |s_1 - s_2| , \quad \textit{for all } \ s_1, s_2 \in \mathbb{R}.$$

If $(H1)$ and $(H2)$ hold, then problem $(P_{\omega,\theta})$ has a unique solution.

Proof. Let (ω_1, θ_1) and (ω_2, θ_2) be two solutions of problem $(\mathrm{P}_{\omega,\theta})$, that is,

$$\begin{aligned} &\frac{\partial \omega_1}{\partial t} - b_1(y,t)\frac{\partial \omega_1}{\partial y} - a_1(l_1(\theta_1))\, b_2(t)\frac{\partial^2 \omega_1}{\partial y^2} = g_1(y,t), \\ &\frac{\partial \theta_1}{\partial t} - b_1(y,t)\frac{\partial \theta_1}{\partial y} - a_2(l_1(\omega_1))\, b_2(t)\frac{\partial^2 \theta_1}{\partial y^2} = g_2(y,t), \end{aligned} \tag{32}$$

with $\omega_1(0,t) = \omega_1(1,t) = 0$, $\theta_1(0,t) = \theta_1(1,t) = 0$ and

$$\begin{aligned} &\frac{\partial \omega_2}{\partial t} - b_1(y,t)\frac{\partial \omega_2}{\partial y} - a_1(l_1(\theta_2))\, b_2(t)\frac{\partial^2 \omega_2}{\partial y^2} = g_1(y,t), \\ &\frac{\partial \theta_2}{\partial t} - b_1(y,t)\frac{\partial \theta_2}{\partial y} - a_2(l_1(\omega_2))\, b_2(t)\frac{\partial^2 \theta_2}{\partial y^2} = g_2(y,t), \end{aligned} \tag{33}$$

with $\omega_2(0,t) = \omega_2(1,t) = 0$ and $\theta_2(0,t) = \theta_2(1,t) = 0$. Subtracting equations (33) from equations (32), we get, respectively,

$$\begin{aligned} &\frac{\partial \omega_1}{\partial t} - \frac{\partial \omega_2}{\partial t} - b_1\left(\frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial y}\right) - a_1(l_1(\theta_1))\, b_2\frac{\partial^2 \omega_1}{\partial y^2} \\ &\qquad\qquad + a_1(l_1(\theta_2))\, b_2\frac{\partial^2 \omega_2}{\partial y^2} = 0, \\ &\frac{\partial \theta_1}{\partial t} - \frac{\partial \theta_2}{\partial t} - b_1\left(\frac{\partial \theta_1}{\partial y} - \frac{\partial \theta_2}{\partial y}\right) - a_2(l_1(\omega_1))\, b_2\frac{\partial^2 \theta_1}{\partial y^2} \\ &\qquad\qquad + a_2(l_1(\omega_2))\, b_2\frac{\partial^2 \theta_2}{\partial y^2} = 0, \end{aligned}$$

and it follows that $(q,r)=(\omega_1-\omega_2,\theta_1-\theta_2)$ is a solution of

$$\begin{aligned}\frac{\partial q}{\partial t}-b_1\frac{\partial q}{\partial y}-a_1\left(l_1(\theta_1)\right)b_2\frac{\partial^2\omega_1}{\partial y^2}+a_1\left(l_1(\theta_2)\right)b_2\frac{\partial^2\omega_2}{\partial y^2}=0,\\ \frac{\partial r}{\partial t}-b_1\frac{\partial r}{\partial y}-a_2\left(l_1(\omega_1)\right)b_2\frac{\partial^2\theta_1}{\partial y^2}+a_2\left(l_1(\omega_2)\right)b_2\frac{\partial^2\theta_2}{\partial y^2}=0,\end{aligned},$$

in $L^2\left(0,T;L^2\left(\Omega\right)\right)$, with $q(0)=0$ and $r(0)=0$. Taking the inner product in $L^2\left(\Omega\right)$, with q in the first equation and with r in the second one, integrating by parts, and adding and subtracting $a_1\left(l_1\left(\theta_1\right)\right)b_2\left(\frac{\partial\omega_2}{\partial y},\frac{\partial q}{\partial y}\right)$ from the first equation, and $a_2\left(l_1\left(\omega_1\right)\right)b_2\left(\frac{\partial\theta_2}{\partial y},\frac{\partial r}{\partial y}\right)$ from the second equation, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|q(t)\right|^2+\frac{1}{2}\frac{\gamma'}{\gamma}\left|q(t)\right|^2+a_1\left(l_1\left(\theta_1\right)\right)b_2\left[\left(\frac{\partial\omega_1}{\partial y},\frac{\partial q}{\partial y}\right)-\left(\frac{\partial\omega_2}{\partial y},\frac{\partial q}{\partial y}\right)\right]\\ =\left[a_1\left(l_1\left(\theta_2\right)\right)-a_1\left(l_1\left(\theta_1\right)\right)\right]b_2\left(\frac{\partial\omega_2}{\partial y},\frac{\partial q}{\partial y}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|r(t)\right|^2+\frac{1}{2}\frac{\gamma'}{\gamma}\left|r(t)\right|^2+a_2\left(l_1\left(\omega_1\right)\right)b_2\left[\left(\frac{\partial\theta_1}{\partial y},\frac{\partial r}{\partial y}\right)-\left(\frac{\partial\theta_2}{\partial y},\frac{\partial r}{\partial y}\right)\right]\\ =\left[a_2\left(l_1\left(\omega_2\right)\right)-a_2\left(l_1\left(\omega_1\right)\right)\right]b_2\left(\frac{\partial\theta_2}{\partial y},\frac{\partial r}{\partial y}\right).\end{aligned}$$

Adding these two equations, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left|q(t)\right|^2+\left|r(t)\right|^2\right)+\frac{1}{2}\frac{\gamma'}{\gamma}\left(\left|q(t)\right|^2+\left|r(t)\right|^2\right)\\ &\quad+a_1\left(l_1\left(\theta_1\right)\right)b_2\left|\frac{\partial q(t)}{\partial y}\right|^2+a_2\left(l_1\left(\omega_1\right)\right)b_2\left|\frac{\partial r(t)}{\partial y}\right|^2\\ &\quad=\left[a_1\left(l_1\left(\theta_2\right)\right)-a_1\left(l_1\left(\theta_1\right)\right)\right]b_2\left(\frac{\partial\omega_2}{\partial y},\frac{\partial q}{\partial y}\right)\\ &\quad+\left[a_2\left(l_1\left(\omega_2\right)\right)-a_2\left(l_1\left(\omega_1\right)\right)\right]b_2\left(\frac{\partial\theta_2}{\partial y},\frac{\partial r}{\partial y}\right).\end{aligned}\tag{34}$$

The third term of the last equation implies that

$$a_1\left(l_1\left(\theta_1\right)\right)b_2\left|\frac{\partial q(t)}{\partial y}\right|^2=a_1\left(l_1\left(\theta_1\right)\right)b_2\left\|q(t)\right\|^2\geq\frac{m_{a_1}}{\gamma_1^2}\left\|q(t)\right\|^2\tag{35}$$

and, similarly, we have

$$a_2\left(l_1\left(\omega_1\right)\right)b_2\left|\frac{\partial r(t)}{\partial y}\right|^2\geq\frac{m_{a_2}}{\gamma_1^2}\left\|r(t)\right\|^2.\tag{36}$$

By the properties of a_i and l_1, using the Schwarz inequality and $(H1)$, one obtains the following upper bounds for the terms on the right hand side of (34) :

$$\begin{aligned} &\frac{A_1}{\gamma_0^2}\left|l_1\left(\theta_2\right)-l_1\left(\theta_1\right)\right|\left|\frac{\partial\omega_2(t)}{\partial y}\right|\left|\frac{\partial q(t)}{\partial y}\right| \leq \frac{A_1c_0}{\gamma_0^2}\left|q(t)\right|\left|\frac{\partial\omega_2(t)}{\partial y}\right|\left\|q(t)\right\|, \\ &\frac{A_2}{\gamma_0^2}\left|l_1\left(\omega_2\right)-l_1\left(\omega_1\right)\right|\left|\frac{\partial\theta_2(t)}{\partial y}\right|\left|\frac{\partial r(t)}{\partial y}\right| \leq \frac{A_2c_0}{\gamma_0^2}\left|r(t)\right|\left|\frac{\partial\theta_2(t)}{\partial y}\right|\left\|r(t)\right\|. \end{aligned} \tag{37}$$

Substituting (35) to (37) in equation (34) and multiplying by 2, similarly to (23), we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\left|q(t)\right|^2+\left|r(t)\right|^2\right)+\left(\frac{2m_a}{\gamma_1^2}-\frac{1}{\varepsilon}\right)\left(\left\|q(t)\right\|^2+\left\|r(t)\right\|^2\right) \\ &\leq \varepsilon\left(\frac{Ac_0}{\gamma_0^2}\right)^2\left(\left|q(t)\right|^2\left|\frac{\partial\omega_2(t)}{\partial y}\right|^2+\left|r(t)\right|^2\left|\frac{\partial\theta_2(t)}{\partial y}\right|^2\right)+\frac{|\gamma'|}{\gamma_0}\left(\left|q(t)\right|^2+\left|r(t)\right|^2\right), \end{aligned}$$

for some $\varepsilon > \gamma_1^2/m_a/2$, where $A = \max\{A_1, A_2\}$. Setting $\varepsilon = \gamma_1^2/m_a$, for example, and integrating from 0 to t in both members of the last inequality, it follows that

$$\left|q(t)\right|^2+\left|r(t)\right|^2 \leq \int_0^t \varphi(s)\left(\left|q(s)\right|^2+\left|r(s)\right|^2\right)ds\,, \tag{38}$$

where the function $\varphi \in L^1(]0,T[)$ is defined by

$$\varphi(s)=\frac{\gamma_1^2}{m_a}\left(\frac{Ac_0}{\gamma_0^2}\right)^2\left(\left|\frac{\partial\omega_2}{\partial y}(s)\right|^2+\left|\frac{\partial\theta_2}{\partial y}(s)\right|^2\right)+\frac{|\gamma'(s)|}{\gamma_0}\,.$$

Finally, by the Gronwall inequality we obtain $\left|q(t)\right|^2+\left|r(t)\right|^2 = 0$, which is equivalent to $q(t) = 0$ and $r(t) = 0$. Thus $\omega_1 = \omega_2$ and $\theta_1 = \theta_2$. □

Now we are in position to prove Theorem 1.

Proof. Let (ω, θ) be the solution of problem $(\mathrm{P}_{\omega,\theta})$ with initial data $\omega_0(y) = u_0\left(\alpha(0)+\gamma(0)y\right)$ and $\theta_0(y) = v_0\left(\alpha(0)+\gamma(0)y\right)$. As $u(x,t) = \omega(y,t)$ and $v(x,t) = \theta(y,t)$, where $x = \alpha(t)+\gamma(t)y$, in order to verify that $(u(x,t), v(x,t))$ given by Theorem 1 is the solution of problem $(\mathrm{P}_{u,v})$, it is sufficient to observe that the transformation $\tau : \hat{Q} \longrightarrow Q$ is a diffeomorphism of class $\mathcal{C}^2$. In fact, by the equalities $u_t = \omega_t - b_1(y,t)\omega_y$ and $u_{xx} = b_2(t)\omega_{yy}$, $v_t = \theta_t - b_1(y,t)\theta_y$ and $v_{xx} = b_2(t)\theta_{yy}$, the existence of a solution for problem $(\mathrm{P}_{\omega,\theta})$ and the regularity of $(\omega(y,t), \theta(y,t))$ given by Theorem 2, we can conclude that $(u(x,t), v(x,t))$ is a solution of $(\mathrm{P}_{\omega,\theta})$. Finally, the uniqueness of the solution for $(\mathrm{P}_{u,v})$ is a consequence of the uniqueness of the solution for $(\mathrm{P}_{\omega,\theta})$, because $u = \omega$ and $v = \theta$. □

5. Exponential decay of the solution

A great number of works have dealt with the weak or regular solutions for parabolic and hyperbolic equations with moving boundaries. In our problem, it brings essential difficulties, because (the geometry of) the domain influences the correctness of the corresponding problem (see [14]).

The goal of this section is to establish a rate of decay for the energy associated to problem $(\mathrm{P}_{u,v})$. Therefore, we obtain the asymptotic behaviour, for a large t, of the natural energy

$$E(t) = \frac{1}{2}\left(|u(t)|^2_{L^2(\Omega_t)} + |v(t)|^2_{L^2(\Omega_t)}\right), \tag{39}$$

inside the time dependent domain $\hat{Q}$. Thus, we can state:

Theorem 4. *Assuming the hypotheses of Theorem 1, if $f_1(x,t) = f_2(x,t) = 0$ in $(P_{u,v})$, then function E satisfies*

$$E(t) \leq E(0)e^{-\delta t}, \quad \textit{for all } t \geq 0, \ \textit{with } \delta > 0.$$

In order to prove this theorem, we need to establish Poincaré's inequality in Ω_t. Thus we have:

Lemma 5. *If $u \in H^1_0(\Omega_t)$ then*

$$|u(t)|^2_{L^2(\Omega_t)} \leq \gamma^2(t)\, |u_x(t)|^2_{L^2(\Omega_t)}.$$

Proof. In fact, from the Fundamental Theorem of Calculus, we have that

$$u(x,t) = \int_{\alpha(t)}^{x} \frac{\partial}{\partial \xi} u(\xi,t) d\xi.$$

From this and Schwarz's inequality, we obtain

$$|u(x,t)|^2_{\mathbb{R}} \leq \gamma(t)\, |u_x(t)|^2_{L^2(\Omega_t)}.$$

Integrating in Ω_t, we get

$$|u(t)|^2_{L^2(\Omega_t)} \leq \gamma^2(t)\, |u_x(t)|^2_{L^2(\Omega_t)}.$$

□

Hence, we are in a position to prove Theorem 4.

Proof. Consider the two differential equations in (2). Taking the inner product in $L^2(\Omega_t)$, when $f_1 = f_2 = 0$, with $u(x,t)$ in the first equation and with $v(x,t)$ in the second one, we have

$$\begin{aligned} &(u_t, u) - a_1(l(v))\,(u_{xx}, u) = 0, \\ &(v_t, v) - a_2(l(u))\,(v_{xx}, v) = 0. \end{aligned} \tag{40}$$

Applying Leibnitz rule and using the null Dirichlet boundary conditions in the first term of each equation in (40), yields

$$\begin{aligned}(u_t, u) &= \frac{1}{2}\frac{d}{dt}\int_{\Omega_t} |u(x,t)|^2\, dx,\\ (v_t, u) &= \frac{1}{2}\frac{d}{dt}\int_{\Omega_t} |v(x,t)|^2\, dx.\end{aligned} \tag{41}$$

Integrating by parts the second term in (40) and from the boundary conditions, we obtain

$$\begin{aligned}(u_{xx}, u) &= -\left|u_x(t)\right|^2_{L^2(\Omega_t)},\\ (v_{xx}, v) &= -\left|v_x(t)\right|^2_{L^2(\Omega_t)}.\end{aligned} \tag{42}$$

Substituting (41) and (42) in (40), adding the two equations and using (39), one gets

$$\frac{d}{dt}E(t) + a_1(l(v))\left|u_x(t)\right|^2_{L^2(\Omega_t)} + a_2(l(u))\left|v_x(t)\right|^2_{L^2(\Omega_t)} = 0\,.$$

By hypothesis $(H5)$, as $m_a = \min\{m_{a_1}, m_{a_2}\}$, we obtain

$$\frac{d}{dt}E(t) + m_a\left(\left|u_x(t)\right|^2_{L^2(\Omega_t)} + \left|v_x(t)\right|^2_{L^2(\Omega_t)}\right) \le 0\,. \tag{43}$$

From the last inequality, we have that $\frac{d}{dt}E(t) \le 0$, for all $t \ge 0$, since $m_a > 0$. So, the energy E is a nonnegative decreasing function.

By Lemma 5 and $(H1)$, we get Poincaré's inequalities

$$\begin{aligned}\left|u(t)\right|^2_{L^2(\Omega_t)} &\le \gamma_1^2\left|u_x(t)\right|^2_{L^2(\Omega_t)},\\ \left|v(t)\right|^2_{L^2(\Omega_t)} &\le \gamma_1^2\left|v_x(t)\right|^2_{L^2(\Omega_t)}.\end{aligned}$$

Thus, from this and the inequality in (43), we obtain

$$\frac{d}{dt}E(t) + \frac{m_a}{\gamma_1^2}\left(\left|u(t)\right|^2_{L^2(\Omega_t)} + \left|v(t)\right|^2_{L^2(\Omega_t)}\right) \le 0\,, \quad \text{for all } t \ge 0\,,$$

and

$$\frac{d}{dt}\left(E(t)e^{\delta t}\right) \le 0\,,$$

where $\delta = 2m_a/\gamma_1^2$. Integrating from 0 to t, we conclude that

$$E(t) \le E(0)\, e^{-\delta t}\,,$$

which proves the exponential decay of the solution when both the reaction forces f_1 and f_2 are null. □

Remark 1: When f_1 and f_2 decay in an appropriate way (see [12]), we can obtain the same result as in Theorem 4 with $f_1(x,t) \neq 0$ and $f_2(x,t) \neq 0$.

Remark 2: The results in Theorems 1 and 4 can be easily generalized to

$$\begin{cases} u_t - a_1 \left(\int_{\Omega_t} v(x,t)dx \right) \Delta u = f_1(x,t)\,, & \text{in } \hat{Q}, \\ v_t - a_2 \left(\int_{\Omega_t} u(x,t)dx \right) \Delta v = f_2(x,t)\,, & \text{in } \hat{Q}, \end{cases}$$

where $\hat{Q} \subset \mathbb{R}^{n+1}$ $(n \geq 1)$ is a bounded non-cylindrical domain defined by

$$\hat{Q} = \bigcup_{0<t<\infty} \Omega_t \times \{t\}\,,$$

with lateral boundary $\hat{\Sigma} = \cup_{0<t<\infty}(\Gamma_t \times \{t\})$. We refer the reader to [19, 13] for further details.

6. Numerical results

In this section we apply a general-purpose Matlab code based on the formulation of MFEM with higher order basis functions to determine the approximate solution of physical problems described by a system of parabolic time-dependent PDEs

$$\boldsymbol{u}_t = A\boldsymbol{u}_{xx} + \boldsymbol{f}\,, \quad x \in \Omega_t\,, \quad t \geq 0\,, \tag{44}$$

where $\boldsymbol{u} = [u_1(x,t), ..., u_n(x,t)]^T$ is the solution vector, under the initial condition $\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x)$ defined on Ω_0. In this mathematical model, the entries of both matrix A and vector $\boldsymbol{f}$ may be functions of $\boldsymbol{u}$, $\partial\boldsymbol{u}/\partial x$ and the independent space and time variables. Problem $(\mathrm{P}_{u,v})$ is a particular case of this system for $n = 2$, where $\boldsymbol{f} = \boldsymbol{f}(x,t)$ and matrix A is diagonal with $a_{i,i} = \int_{\Omega_t} u_i(x,t)dx$. We should point out that our numerical algorithm allows us to solve the problem immediately in the non cylindrical domain. Below, we present a brief description of some fundamental aspects of our formulation of the MFEM.

6.1. Formulation of the method

To construct the semi-discrete approximation, we consider an initial independent partition of Ω_0 associate with each dependent variable u_m of (44). So, we define the $N_m - 1$ interior space nodes, of the mth grid

$$\mathcal{G}_m \quad : \quad \alpha(0) < X_{m,2} < \ldots < X_{m,N_m} < \beta(0)\,.$$

The main characteristic of the MFEM is the mobility of grid nodes allowing the adaptivity of the spatial mesh. These are treated as unknown time-dependent variables which must be evaluated as part of the solution procedure. Therefore, the length of each finite element

$$\Omega_{m,e}(t) = [X_{m,e}(t), X_{m,e+1}(t)]\,, \quad t \geq 0\,,$$

vary continuously with time so that the solution becomes suitably represented. To solve efficiently problems with moving boundaries, a special boundary technique is developed by the introduction of two nodes $X_{m,1}(t)$ and $X_{m,N_m+1}(t)$ describing the position of the moving ends of Ω_t at each instant t. Approximating the mth dependent variable on the canonical element $\Omega_{m,e}$ by a polynomial of degree r, we may write

$$U_{m,e}(x,t) = \sum_{j=1}^{r+1} u(x^j_{m,e}, t)\phi^j_{m,e}(x)\,, \tag{45}$$

where $x^j_{m,e}$ is the jth interpolation point of $\Omega_{m,e}(t)$ and $\{\phi^j_{m,e}(x)\}_{j=1,\ldots,r+1}$ represents the local polynomial basis functions. The arbitrary degree r is chosen by the user and might be different in two adjacent elements of the mth grid then, in general, we have $r = r(m,e) > 1$. The interpolating points are defined as in [1], in such a way that to minimize the maximum absolute error of the local approximation.

In the development of numerical algorithm, the mth component of the numerical approximation $\boldsymbol{U}$ of $\boldsymbol{u}$ is smoothed in a neighbourhood of each $X_{m,e}$ through cubic Hermite polynomials, in such way that its possible to define approximations of spatial derivatives at spatial nodes. $U_m(\cdot,t)$ is chosen to be continuous function on Ω_t and depends on nodal amplitudes and nodal positions of attached grid. Let $\{\xi^j_m(t) : j = 1,2,\ldots,\tilde{N}_m\}$ be the set of the global interpolation points in Ω_t. Therefore, for a fixed time $t > 0$, we define the interpolant U_m by

$$U_m(x,t) = \boldsymbol{\Psi}^{(m)}\,\boldsymbol{U}^{(m)}\,, \tag{46}$$

where $\boldsymbol{U}^{(m)} = \left(u_m(\xi^1_m),\ldots,u_m(\xi^{\tilde{N}_m}_m)\right)^T$ and $\boldsymbol{\Psi}^{(m)} = \left(\Phi^1_m(x),\ldots,\Phi^{\tilde{N}_m}_m(x)\right)$ represents the global polynomial basis functions.

The solution of PDE system is obtained by minimizing the sum of square L^2-norm of discretized residuals in order to the time derivatives of the solutions on the interpolation nodes and nodal velocities, that is,

$$\min_{\dot{U}^j_m,\,\dot{X}_{m,e}} \sum_{m=1}^{n} \left(\int_{\Omega_t} \mathcal{R}^2_m dx + P_m\right)\,, \tag{47}$$

where $(\dot{\ }) = d/dt$ and $\mathcal{R}_m$ is the residual of the mth discretized PDE. Spatial discretization gives origin to a system of ordinary differential equations (ODE), that can degenerate when singularities occur. These can be of two kinds: parallelism and nodal coalescence. While nodal coalescence is overcome by fixing the minimum internodal distance allowed, parallelism is dealt by the addition of a penalty term to the objective function in (47).

The MFEM is a continuously moving grid method, where the node movement and PDE integration are fully coupled. By the minimization process, each of the PDEs from (44) generates a system of ODEs. The semi-discrete equations resulting from the application of MFEM to the mathematical model given by

(44) are combined with the boundary conditions to yield a system of ODEs that can be reordered to matrix form

$$M(t,\, \boldsymbol{Y})\, \dot{\boldsymbol{Y}} = \boldsymbol{F}(t,\, \boldsymbol{Y})\,. \tag{48}$$

Our formulation originates sparse mass matrices strongly dependent on $\boldsymbol{Y}$. Nodal amplitudes and nodal positions are found interlaced in vector $\boldsymbol{Y}$, ordered in such way that M is a quasi-diagonal block matrix. The solution of the initial value problem (48) can be obtained by an appropriate ODE integrator. In the present case, we use the function *ode15s* from Matlab ode suite [23], a variable order variable time-step ODE integrator for stiff problems. We included this function in our Matlab code, exploited the sparsity of the mass matrix M and selected the numerical differential formulae methods to perform the integration. The details of the interface that implements this spatial discretization and applications of the MFEM can be found in [18, 8, 22] and the references therein.

6.2. Numerical simulation

The numerical results presented here are obtained in the Matlab environment using a computer with an Intel Core i7 − 3960X processor at 3.30 GHz. We compute all the integrals without truncation error using Lobatto's quadrature and use the existing standard values of the optional user-modifier method parameters, such as the minimal node distance allowed or the ODE solver tolerances. Below we present some examples of dilations γ that satisfy the hypotheses $(H1)$ and $(H2)$ about the domain $\hat{Q}$.

We assume that $\gamma(t) = \beta(t) - \alpha(t)$, $-\alpha'(t) > 0$ and $\beta'(t) > 0$. So, considering a limited variation (say by K) of the position of both moving boundaries, we must have

$$0 < -\alpha'(t) \leq K\,, \quad 0 < \beta'(t) \leq K\,, \quad \texttt{for all} t \geq 0\,. \tag{49}$$

Let

$$\left| \begin{array}{ll} \alpha(t) = \alpha(0) - \alpha_1 \Psi_i(t)\,, & 0 < \alpha_1, \\ \beta(t) = \beta(0) + \beta_1 \Psi_j(t)\,, & 0 < \beta_1, \end{array} \right. \tag{50}$$

where the indices i and j may be equal or not.

Examples of functions Ψ.

1. Integrating each of inequalities (49) from 0 to t, we obtain $\hat{Q}$ defined by (50) with

$$\Psi_1(t) = t\,, \quad \alpha_1 \leq K\,, \quad \beta_1 \leq K\,.$$

Note that, in this case, we have linear boundaries.

2. The lateral boundary of $\hat{Q}$ given by

$$\Psi_2(t) = (t + t_0)^{1/n} - (t_0)^{1/n}\ ,$$

for $n = 2, 3, ...$ and $t_0 = \left(\frac{\alpha_1}{nK}\right)^{\frac{n}{n-1}}$, or $t_0 = \left(\frac{\beta_1}{nK}\right)^{\frac{n}{n-1}}$, is not linear.

3. This example shows that when $t \to \infty$ the domain $\hat{Q}$ is asymptotic to a cylinder

$$\Psi_3(t) = \frac{1}{\sqrt[n]{t_0}} - \frac{1}{\sqrt[n]{t+t_0}}\,,$$

with $\alpha_1 = \beta_1 = 1$ and $1/t_0 = (nK)^{n/(3n-1)}$.

Example 1: Let $\hat{Q}$ be defined by

$$\alpha(t) = \alpha(0) - \Psi_3(t)\,, \quad \beta(t) = \beta(0) + \Psi_3(t)\,, \quad t \geq 0\,,$$

with $n = 2$ and $t_0 = 1$. In order to illustrate the asymptotic behaviour of the solutions under the assumptions $(H3)-(H5)$, we consider system (2) in $\hat{Q}$, with $\Omega_0 =]0,1[$,

$$a_1(s) = 1 + \sin(2s)\,, \quad a_2(s) = 1 + \frac{1}{1+s^2}\,,$$

and

$$f_1(x,t) = \frac{0.1e^{-x}}{(1+t)^{10}}\,, \quad f_2(x,t) = \frac{0.01x}{(1+t)^2}\,.$$

Hypothesis $(H5)$ is satisfied with $m_a = 0$, $M_a = \max\{M_{a_1}, M_{a_2}\} = 2$ and the following Lipschitz constant $A \geq \max\{A_1, A_2\} = 2$. Moreover, the external sources f_1 and f_2 satisfy $(H4)$. We perform the numerical simulation using the initial conditions

$$u_0(x) = 32(\alpha(0) - x)(x - \beta(0)) \;\text{ and }\; v_0(x) = 1 - \cos(4\pi x)\,,$$

which satisfy $(H3)$. Initially, the population of the first species is concentrated in the middle of Ω_0 and the individuals of the second kind constitute two clusters, each one in half of the spatial domain.

We computed the numerical solution at several times from $t = 0$ to $t = 1$, for both the dependent variables, with local polynomial approximations of degree 5 in each of the four finite elements used. Initially, nodes are placed forming uniform grids in Ω_0.

In Figure 1, we observe the extinction of both species which is consistent with the exponential decay of the solution. Before that occurs, the behaviour of the solutions is different: we see a high rate of decay of the density of the population (the death of individuals) of the first specie, without significant diffusion in Figure 1 (left), however, in the first instants, Figure 1 (right) shows the mobility of a large number of individuals of the second kind to the central region.

To display better the regularity of the solution in the neighborhood of the moving boundaries, in Figure 2 (left), we present a zooming of profiles for the dependent variable u, centered at $(\beta(t), 0)$. In figure 2 (right) we plotted $X_{2,j}(t)$ versus $v(X_{2,j}(t), t)$, for $0 \leq t \leq 1$. We observe the descendending trajectories of the interior separation nodes and that the evolution of the adaptive mesh processes with smoothness even though the initial situation where the solution has sharp variations.

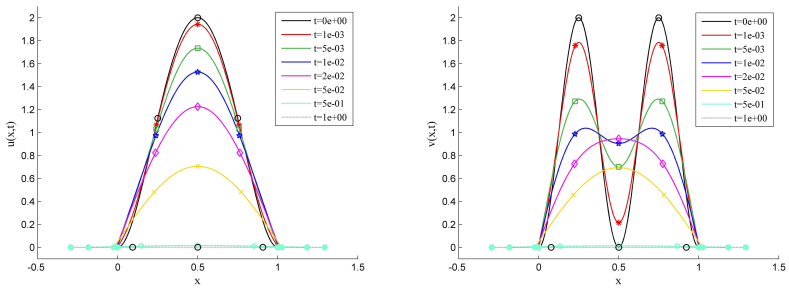

Figure 1: Solution profiles at different times for u (left) and for v (right)

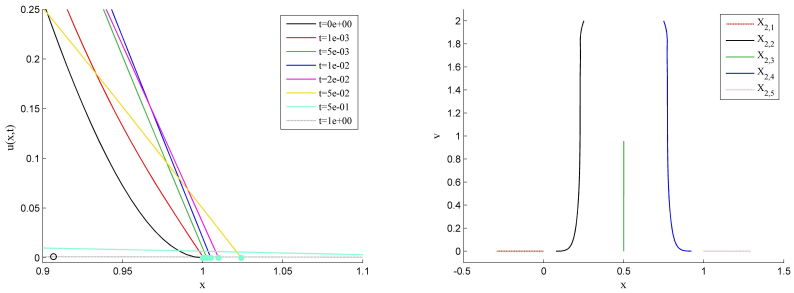

Figure 2: Zooming of Figure 1 on a region centered in $(\beta(t), 0)$, for u (left) and trajectories of the separation nodes on the mesh associated to v (right)

As we use a high degree local approximation, the MFEM does not have to relocate them quickly. We have similar smooth trajectories of grid nodes associated to u.

The approximate numerical solution is shown in the 3D graph of Figure 3. We observe the decline of solutions u (left) and v (right), with the increasing of time. Finally, in Figure 4, we represent the history of values u (left) and v (right) for two fixed points of spatial domain, $x_1 = 0.25$ and $x_2 = 0.5$. We see a high rate of decay of values $u(x_k, t)$ and $v(x_k, t)$, $k = 1, 2$, as t increase, leading to an exponential energy decay of solutions.

Example 2: In this second example we use a different limited dilation γ defined by Ψ_2, with $n = 3$, $K = 1/2$ and $\alpha_1 = 1$, that is

$$\alpha(t) = \sqrt{2/3} - \sqrt[3]{t + (2/3)^{3/2}}\,, \quad \beta(t) = 1 - \alpha(t)\,, \quad t \geq 0\,.$$

We want to illustrate that the exponential decay of the solution depends on an appropriate rate of decay of functions f_i, $i = 1, 2$. So, we consider

$$a_1(s) = 2 - \frac{1}{1 + s^2}\,, \quad a_2(s) = e^{-s^2}\,,$$

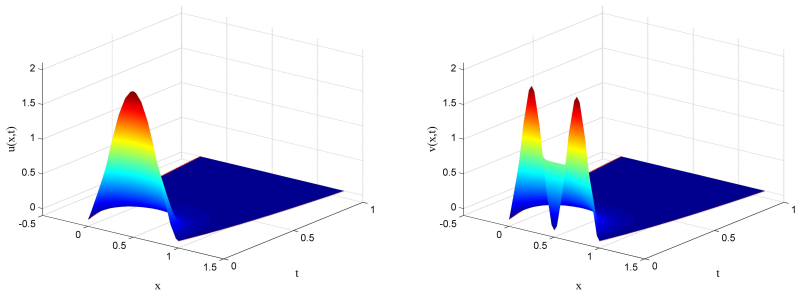

Figure 3: Evolution of the approximate solution u (left) and v (right) in space-time domain

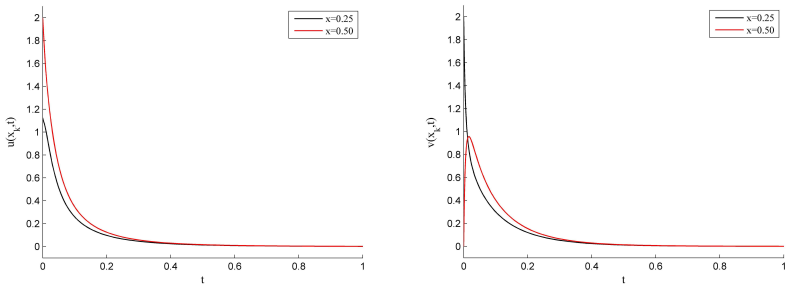

Figure 4: Asymptotic behaviour of the solutions u (left) and v (right)

and the reaction forces

$$f_1(x,t) = \frac{0.1x}{(1+t)^4} \quad \text{and} \quad f_2(x,t) = \frac{e^{-x^2}}{(1+t)^6}\,. \tag{51}$$

Initial conditions are given by $u(x,0) = \mathcal{S}_3(x)$ and $v(x,0) = \bar{\mathcal{S}}_3(x)$, where $\mathcal{S}_3$, $\bar{\mathcal{S}}_3$ represent the natural spline functions of degree three, that interpolate the points of coordinates

$$\{(0,0),(0.2,1),(0.5,0.5),(1,0)\} \text{ and } \{(0,0),(0.6,0.65),(0.8,1),(1,0)\},$$

respectively. This data satisfies the hypotheses of Theorem 1.

Numerical simulations are carried out using independent initial grids with five points. According initial conditions we have concentrated the interior separation nodes on the first half of spatial domain, for u, and near the right end of Ω_0, for v. The initial partitions of Ω_0 are determined by the points 0, 0.1, 0.25, 0.5 and 1, for the grid associated to u and by 0, 0.5, 0.75, 0.9 and 1, for the grid associated to v. The MFEM solution were obtained with locally approximations of degree four in each finite element of both grids. The integration time interval considered is $[0,1]$.

The distribution of the two population densities, in the non cylindrical domain $\hat{Q}$ for $t \leq 1$, are shown in Figure 5. As expected, we observe that the

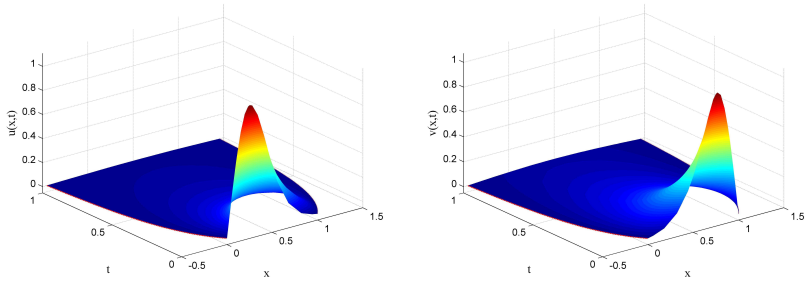

Figure 5: Approximate solution u (left) and v (right) in $\Omega_t \times]0,1[$

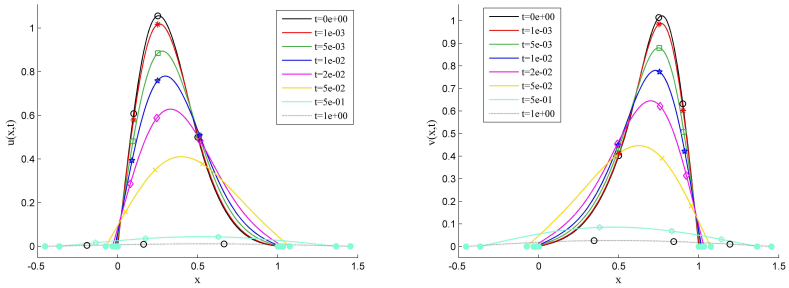

Figure 6: Evolution of population densities at different times, u (left) and v (right)

extinction of population of both species occurs in a finite time. These results demonstrate that the MFEM can produce accurate results efficiently using a reduced number of nodes as well as calculation time. In Figure 6, we see the results obtained for $u(x, t_k)$ (left) and $v(x, t_k)$ (right), computed for various instants t_k. We observed similar behaviours of two species: a decrease in both populations in time, together with a more uniform redistribution in space of each specie, due to the mobility of a large number of individuals.

The MFEM automatically relocates moving nodes in order to concentrate them in regions where the solution has sharp profiles. As we use a five degree local approximation the method is able to move nodes with smoothness. This can be seen in Figure 7 were we present the movement of two boundaries and the interior separation nodes associated to u. We have an analogous regular smooth movement for the grid associated to v.

Finally, in Figure 8, we plotted the dependent variable v versus time for a two different reaction forces associated to the second PDE: f_2 defined in (51) and a different function $\bar{f}_2(x,t) = te^{-x^2}$, at two fixed values of spatial variable x. It is observed that (see Figure 8 (left)), the asymptotic decay of energy for f_2. On the contrary, in Figure 8 (right), we see that function $\bar{f}_2$ does not have an appropriate decay leading to an asymptotic behaviour of the solution.

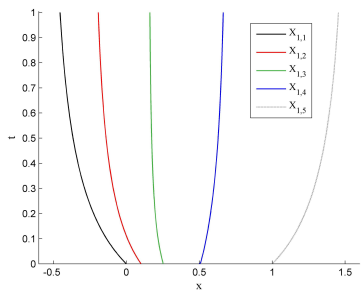

Figure 7: Mesh movement, associated to u

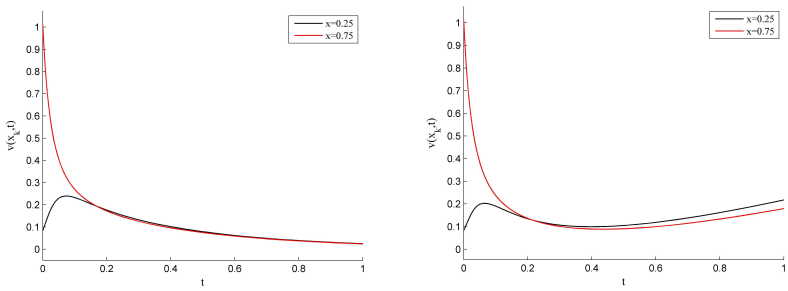

Figure 8: History of $v(x,t)$ at two fixed values of x for the functions f_2 (left) and $\bar{f}_2$ (right)

7. Conclusions

We prove the existence and uniqueness of strong global solutions for a large class of nonlocal nonlinear coupled systems with moving boundaries. Moreover, we show the exponential decay of the solutions. By our numerical algorithm, based on the MFEM with piecewise polynomial of arbitrary degree basis functions in space, we are able to solve the initial problem without using the transformation in the cylindrical domain. Two numerical experiments were presented considering different dilations γ, to show the moving boundary for the problem and the dependence of the exponential decay of the solutions on the functions f_i, $i = 1, 2$. The numerical results demonstrate the accuracy and robustness of our Matlab code based on the MFEM; in particular, they are in agreement with the asymptotic behaviour of the analytic solution. The application of Euler-Galerkin finite element method to establish an error estimate of solutions for our problem is in progress.

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